Addition Operations on The Monadic Theories of Finite Structures

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by

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1. Abstract

The results presented in this thesis are organized in two parts, both dealing with the logical properties of finite structures. In particular we will deal with the first order and monadic theories of finite structures. The first part deals with classes of finite structures constructible by addition operations on the monadic theories of the structures. The second part deals with the first order theory of random graphs, and in particular with necessary and sufficient conditions for the holding of the hereditary 0-1 law in the graph.

1.1. Constructible classes. In this part we deal with three kinds of classes of finite structures: Classes constructible by addition operation, classes with bounded $m$-ary patch-width, and classes monadically interpretable in trees. Our interest in this subject started from investigating spectra of monadic sentences, so let us begin with a short description of spectra. Let $\phi$ be a sentence in (a fragment of) second order logic (SOL). The spectrum of $\phi$ is the set $\{n \in \mathbb{N} : \phi \text{ has a model of size } n\}$.

In 1952 Scholz defined the notion of spectrum and asked for a characterization of all spectra of first order (FO) sentences. In [1] Asser asked if the complement of a FO spectrum is itself a FO spectrum.

Definition 1.1. A set $A \subseteq \mathbb{N}$ is eventually periodic if for some $n, p \in \mathbb{N}$, for all $m > n$, $m \in A$ if and only if $m + p \in A$.

In [8] Durand, Fagin and Loescher showed that the spectrum of a FO sentence in a vocabulary with finitely many unary relation symbols and one function symbol is eventually periodic. In [12] Gurevich and Shelah generalized this for spectrum of monadic second order (MSO) sentence in the same vocabulary. Inspired by [12] Fisher and Makowsky in [10] showed that the spectrum of a CMSO sentence (a monadic sentence with counting quantifiers) is eventually periodic provided that all its models have bounded patch-width. A many sorted version for the context of graphs is the generalization of the Parikh’s theorem proved by Courcelle in [7]. The notion of patch-width of structures (usually graphs) is a complexity measure on structures, generalizing clique-width. The proofs of [10] remains valid if we consider $m$-ary patch-width, i.e. we allow $m$-ary relations as auxiliary relations. In [14] Shelah generalized the proof of [12] and showed eventual periodicity for a MSO sentence provided that all its models are constructible by recursion using operations that preserve monadic theory. A similar result is true for classes monadically interpretable in trees.

The above results on eventual periodicity led us to ask: What are the relations between the different notions for which we have eventual periodicity of MSO spectra? In other words do we have three different results, or are they all equivalent? In [5] Courcelle proved (using somewhat different notations) that a class of structures is constructible if and only if it is monadically interpretable in trees, thus implying that two of the results coincide. In section 5 we give a proof of Courcelle’s result more coherent with our definitions, which we use later on. We prove that the notions of bounded $m$-ary patch-width is very close to $m$-constructibility (constructibility where we allow $m$-ary relations as auxiliary relations). In section 6 we show that for $m \geq 3$ a class of models is contained in a $m$-constructible class if and only if it is contained in a 3-constructible class. The main result in this part in therefore:
Theorem 1.2. Let $\mathcal{R}$ be a class of $\tau$-structures. Then $\mathcal{R}$ is contained in a $m$-constructible class for some $m \in \mathbb{N}$ if and only if $\mathcal{R}$ is contained in a 3-constructible class.

The same holds for classes of bounded $m$-ary patch-width. Finally, in section 7, we show that in the above theorem we cannot replace 3-constructible by 1-constructible. That is:

Theorem 1.3. There exists a vocabulary $\tau$, and a class of $\tau$-structures $\mathcal{R}$, contained in some 3-constructible class, that is not contained in any 1-constructible class.

We give a specific example. The case $m = 2$ is left open.

The results presented in this part were published in [15].

1.2. Hereditary zero-one laws. In this part we will investigate the random graph on the set $[n] = \{1, 2, ..., n\}$ were the probability of a pair $i \neq j \in [n]$ being connected by an edge depends only on their distance $|i - j|$. Let us define:

Definition 1.4. For a sequence $\bar{p} = (p_1, p_2, p_3, ...) \forall p_i$ is a probability i.e. a real in $[0, 1]$, let $M^n_{\bar{p}}$ be the random graph defined by:

- The set of vertices is $[n] = \{1, 2, ..., n\}$.
- For $i, j \leq n, i \neq j$ the probability of $\{i, j\}$ being an edge is $p_{|i-j|}$.
- All the edges are drawn independently.

If $\mathcal{L}$ is some logic, we say that $M^n_{\bar{p}}$ satisfies the 0-1 law for the logic $\mathcal{L}$ if for each sentence $\psi \in \mathcal{L}$ the probability that $\psi$ holds in $M^n_{\bar{p}}$ tends to 0 or 1, as $n$ approaches $\infty$. The relations between properties of $\bar{p}$ and the asymptotic behavior of $M^n_{\bar{p}}$ were investigated in [13]. It was proved there that for $\mathcal{L}$, the first order logic in the vocabulary with only the adjacency relation, we have:

Theorem 1.5. (1) Assume $\bar{p} = (p_1, p_2, ...) \exists 0 \leq p_i < 1 \forall i > 0$ and let $f_\bar{p}(n) := \log(\prod_{i=1}^{n}(1-p_i))/\log(n)$. If $\lim_{n \to \infty} f_\bar{p}(n) = 0$ then $M^n_{\bar{p}}$ satisfies the 0-1 law for $\mathcal{L}$.

(2) The demand above on $f_\bar{p}$ is the best possible. Formally for each $\epsilon > 0$, there exists some $\tilde{p}$ with $0 \leq p_i < 1 \forall i > 0$ such that $f_\tilde{p}(n) < \epsilon$ but the 0-1 law fails for $M^n_{\bar{p}}$.

Some examples of the failure of the 0-1 law, and specifically the construction for part (2) above, are given in [13] by either adding zeros to a given sequence or decreasing some of the members of a given sequence. Part (1) above gives a necessary condition on $\bar{p}$ for the 0-1 law to hold in $M^n_{\bar{p}}$, but the condition is not sufficient and a full characterization of $\bar{p}$ seems to be harder. However, we give below a complete characterization of $\bar{p}$ in terms of the 0-1 law in $M^n_q$ for all $q \approx$ dominated by $\bar{p}$, in the appropriate sense. Alternatively one may ask which of the asymptotic properties of $M^n_{\bar{p}}$ are kept under the operations on $\bar{p}$ mentioned above. We deal with this question here. Define:

Definition 1.6. For a sequence $\bar{p} = (p_1, p_2, ...)$:

1. $Gen_1(\bar{p})$ is the set of all sequences $\bar{q} = (q_1, q_2, ...)$ obtained from $\bar{p}$ by adding zeros to $\bar{p}$. Formally $\bar{q} \in Gen_1(\bar{p})$ iff for some increasing $F : \mathbb{N} \to \mathbb{N}$ we have for all $l > 0$

$$q_l = \begin{cases} p_i & F(i) = l \\ 0 & l \notin Im(F) \end{cases}$$
(2) \( \text{Gen}_2(\bar{p}) := \{q = (q_1, q_2, \ldots) : l > 0 \Rightarrow q_l \in [0, p_l]\} \).

(3) \( \text{Gen}_3(\bar{p}) := \{q = (q_1, q_2, \ldots) : l > 0 \Rightarrow q_l \in \{0, p_l\}\} \).

**Definition 1.7.** Let \( \bar{p} = (p_1, p_2, \ldots) \) be a sequence of probabilities and \( \mathcal{L} \) be some logic. For a sentence \( \psi \in \mathcal{L} \) denote by \( \Pr[M_n^\bar{p} \models \psi] \) the probability that \( \psi \) holds in \( M_n^\bar{p} \).

1. We say that \( M_n^\bar{p} \) satisfies the 0-1 law for \( \mathcal{L} \), if for all \( \psi \in \mathcal{L} \) the limit \( \lim_{n \to \infty} \Pr[M_n^\bar{p} \models \psi] \) exists and belongs to \( \{0, 1\} \).
2. We say that \( M_n^\bar{p} \) satisfies the convergence law for \( \mathcal{L} \), if for all \( \psi \in \mathcal{L} \) the limit \( \lim_{n \to \infty} \Pr[M_n^\bar{p} \models \psi] \) exists.
3. We say that \( M_n^\bar{p} \) satisfies the weak convergence law for \( \mathcal{L} \), if for all \( \psi \in \mathcal{L} \), \( \limsup_{n \to \infty} \Pr[M_n^\bar{p} \models \psi] - \liminf_{n \to \infty} \Pr[M_n^\bar{p} \models \psi] < 1 \).
4. For \( i \in \{1, 2, 3\} \) we say that \( \bar{p} \) \( i \)-hereditarily satisfies the 0-1 law for \( \mathcal{L} \), if for all \( q \in \text{Gen}_i(\bar{p}) \), \( M_n^q \) satisfies the 0-1 law for \( \mathcal{L} \).
5. Similarly to (4) for the convergence and weak convergence law.

The main theorem of this paper is the following strengthening of theorem 1.5:

**Theorem 1.8.** Let \( \bar{p} = (p_1, p_2, \ldots) \) be such that 0 \( \leq p_i < 1 \) for all \( i > 0 \), and \( j \in \{1, 2, 3\} \). Then \( \bar{p} \) \( j \)-hereditarily satisfies the 0-1 law for \( \mathcal{L} \) iff

\[
(*) \quad \lim_{n \to \infty} \log(n) \frac{\prod_{i=1}^{n}(1 - p_i))}{\log n} = 0.
\]

Moreover we may replace above the "0-1 law" by the "convergence law" or "weak convergence law".

Note that the 0-1 law implies the convergence law which in turn implies the weak convergence law. Hence it is enough to prove the "only if" direction for the 0-1 law and the "if" direction for the weak convergence law. Also note that the "only if" direction is an immediate conclusion of Theorem 1.5. The case \( j = 1 \) of 1.8 is proved in section 10, and the case \( j \in \{2, 3\} \) is proved in section 11. In section 12 we deal with the case \( U^* = U^*(\bar{p}) := \{i : p_i = 1\} \) is not empty. We give an almost full analysis of the hereditary 0 – 1 law in this case as well. The only case which is not fully characterized is the case \( j = 1 \) and \( |U^*(\bar{p})| = 1 \). We give some results regarding this case in section 13. The following table summarizes the results in this article regarding the \( j \)-hereditary laws.

| \( |U^*| \) | \( 2 \leq |U^*| < \infty \) | \( |U^*| = 1 \) | \( |U^*| = 0 \) |
|---|---|---|---|
| \( j = 1 \) | The 0-1 law holds | See section, \( \{l : 0 < p_l < 1\} = \emptyset \) | \( \lim_{n \to \infty} \frac{\log(n) \prod_{l=1}^{n}(1 - p_l))}{\log n} = 0 \) |
| \( j = 2 \) | The 0-1 law holds \( \uparrow \) | \( |\{l : p_l > 0\}| \leq 1 \) | The 0-1 law holds \( \uparrow \) |
| \( j = 3 \) | The 0-1 law holds \( \uparrow \) | \( \{l : 0 < p_l < 1\} = \emptyset \) | The weak convergence law holds \( \uparrow \) |
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